# Formulas of the Continuum of Real Numbers

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**Abstract**

This paper investigates some properties of two positional number systems — binary (base-2) and decimal. These properties are compared. The well-known formula of the continuum expressed in the binary number system is examined. A formula describing the continuum in the decimal number system is established.

## 1. Introduction

**1.1.1** As an introduction, we present two quotes from the monograph [1] Paul J. Cohen. *Set theory and the continum hypothesis.* 1966 (p. 67).

> "Let $C$ denote the set $P(w)$. We know $\overline{\overline{C}}$ is a uncountable set. Cantor considered the question of determining the position of $\overline{\overline{C}}$ in the transfinite series of the $\boldsymbol{\aleph_\alpha}$ [1]). Since $\overline{\overline{\boldsymbol{C}}} > \boldsymbol{\aleph_0}$ the simplest conjective is
>
> CANTOR'S CONTINUUM HYPOTHESIS. $\overline{\overline{\boldsymbol{C}}} = \boldsymbol{\aleph_1}$."

ibid., pp. 98–99:

> "Cantor's theorem says $\overline{\overline{P(a)}} > \overline{\overline{a}}$ so $\overline{\overline{P(a)}} = \overline{\overline{\beta}}$, which is exactly the GCH. It is more usually written $2^{\aleph_\alpha} = \aleph_{a+1}$ which means precisely the same."

(GCH — *generalized continuum hypothesis*)

## 2. Properties of linear sequences of digits in the decimal number system

### 2.1.1 In this work, we examine only positive real numbers.

Let us present Table 1 (see Appendix). It shows the irrational number $\boldsymbol{\sqrt{2}}$. The *non-integer part* of this number is written in the table with 400 digits (**here and hereafter, the term "digits" will denote decimal digits from 0 to 9**). Using these 400 digits (decimal digits), we construct Table 2 (see Appendix).

**2.1.2** In this table, at the front, *"vertically"*, the numbers of **integers** are indicated: starting with the number 1 and then in order: $2, 3, 4, 5, \dots$ "*Horizontally*" (***i.e., in the form of rows***) the digits that make up the *non-integer part* of the number $\boldsymbol{\sqrt{2}}$ are presented.

**2.1.3** Each row contains 20 digits; at the top (above Table 2), the digit numbers are indicated. These 20 digits were sequentially transferred from Table 1 to Table 2.

**2.1.4** All rows (i.e., the ***whole real numbers*** written in them) are constructed as follows:

first number: 4142135623... ("*row N1*")

second number: 1688724209... ("*row N2*")

third number: 7187537694... ("*row N3*")

and so on.

**2.2.1** Let us choose 20 integer (positive) numbers, each constructed from 20 digits, in various ways. We applied the method described in §2.1.1–§2.1.3 so that each row in Table 2 can be considered both as a record of some integer (positive) number and ***as some random linear combination of the 10 digits (decimal digits):*** $\mathbf{0, 1, 2, 3, 4, 5, 6, 7, 8, 9}$.

**2.2.2 According to this approach, in Table 2, there are 20 different linear combinations of the ten given digits.**

**2.3.1 We pose the general problem** (within the limits of linear combinations of $\leq 20$ digits): *to specify the number of possible combinations (i.e., the number of different integers).* We will denote this number by $N$:

- numbers consisting of only one digit, i.e., written with one digit (1, or 2, or 3, ...or 8, or 9)

$$N = 0.9(10^1);$$

- numbers constructed from two digits (10, or 11, or 12, ...or 98, or 99)

$$N = 10^2;$$

  (we introduced — *conditionally* — into two-digit numbers also such combinations: $01, 02, 03, \ldots$);

- numbers constructed from three digits (100, or 101, or 102, ...or 998, or 999, as well as combinations: $001, 002, 003, \ldots, 011, 012, 013, \ldots$)

$$N = 10^3;$$

- numbers constructed from four digits (1000, or 1001, or 1002, ...or 9998, or 9999, as well as combinations:
  $0001, 0002, 0003, \ldots, 0011, 0012, 0013 \ldots 0110, 0111, 0112, \ldots$)

$$N = 10^4;$$

  and so on.

**2.3.2** Let us denote the above exponent $(1, 2, 3, 4, \ldots)$ by $n$. **All integers represented as rows in Table 2 are written with 20 digits (decimal digits); this means that**

$$n = 20, \quad N = 10^{20}$$

## 3. Some differences between two number systems — binary and decimal

**3.1.1** In Section 2, we examined numbers represented in the decimal number system. Let us also examine numbers written according to the binary number system.

**3.1.2 Both systems for writing real numbers**, currently in use — *binary and decimal* — **are positional; however, there are *radical differences* between them**.

**3.2.1 In the binary number system**, real numbers are written with two digits — zero and one. Let us investigate such a notation of numbers using one of the elements of the first row of table 3 as an example. (Each element of this table denotes one of the digits: $0, 1, 2, 3, \ldots 7, 8, 9$; each row is a record of some real number.)

$$a_{1,1} a_{1,2} a_{1,4} a_{1,5} a_{1,6} a_{1,7} a_{1,8} a_{1,9} a_{1,10} \cdots$$

**3.2.2** Let us analyze the values of any element, for example $a_{1,1}$, written according to the binary system

- if $a_{1,1} \neq 0$, then it means $a_{1,1} = 1$;
- if $a_{1,1} \neq 1$, then it means $a_{1,1} = 0$

**3.3.1 We state.**

*The binary notation system of real numbers excludes any random change (free variation) in the values of the element $a_{1,1}$* (and also the values of all other elements of Table 3).

**3.3.2** For comparison, let us analyze the values of the element $a_{1,1}$ when using **digits of the decimal number system**.

**3.4.1** Let us limit ourselves to three digits — $0, 1, 3$:

$$\text{if } a_{1,1} \neq 0, \quad \text{then it means } a_{1,1} = ?$$

Let us limit ourselves to four digits — $0, 1, 3, 5$:

$$\text{if } a_{1,1} \neq 0, \quad \text{then it means } a_{1,1} = ??$$

and so on.

**3.4.2 We state.**

*Starting from three digits used for writing real numbers, there is a free variation in the values of the element* $a_{1,1}$.

**3.5.1** In connection with what is presented in §3.2.2–§3.4.2, let us consider integer (positive) numbers. Let us choose, for example, numbers that must be represented by *finite linear sequences*, **consisting of 5 digits**.

**3.5.2** We will write these integer (positive) numbers **according to the decimal number system**. We will obtain $\boldsymbol{\sim 10^5}$ ***combinations*** of the composition of the specified sequences constructed from 5 digits, i.e., we will obtain records of $\boldsymbol{\sim 10^5}$ ***different numbers***.

**3.5.3** If we use the **binary number system**, then in the form of linear sequences constructed from two digits, we can write only $\boldsymbol{\sim 2^5}$ ***integers***, i.e., form $\boldsymbol{\sim 2^5}$ ***different combinations***, consisting of 5 digits (0 or 1).

**3.5.4** Let us provide the following examples of writing integers:

**for $\boldsymbol{n = 5}$** (sequence of 5 digits)

$$10^5 \lesssim 100000 \text{ numbers}; \qquad 2^5 \lesssim 32 \text{ numbers};$$

**for $\boldsymbol{n = 7}$** (sequence of 7 digits)

$$10^7 \lesssim 10000000 \text{ numbers}; \qquad 2^7 \lesssim 128 \text{ numbers};$$

**for $\boldsymbol{n = 9}$** (sequence of 9 digits)

$$10^9 \lesssim 1 \textbf{ billion numbers}; \qquad 2^9 \lesssim 512 \text{ numbers};$$

**for $\boldsymbol{n = 20}$** (see sequences of digits in Tables 2 and 4)

$$10^{20} \text{ numbers}; \qquad 2^{20} \lesssim 1048576 \text{ numbers}$$

**3.6.1** Let us choose some integers. Let us represent the selected numbers in the decimal and binary number systems.

| | | | | |
|---|---|---|---|---|
| 3 | 30 | 300 | 3000 | 30000 |
| 11; | 11110; | 100101100; | 101110111000; | 111010100110000 |

**3.6.2** The number 30000 has five digits. In the binary system, 15 digits are needed to write this number; using $\leq 5$ digits — 0 and 1 — we can write only $\lesssim 32$ integers.

**3.6.3** As noted in §3.1.2, both the binary and decimal number systems are positional, but there are *radical differences* between them. One of these differences was examined in §3.2.2–§3.5.4. The above **pairs of integers written in different systems indicate another difference**:

*using only two digits (*0 *and* 1*) leads to an accelerated increase in the number of positions (digits) in writing of integers compared to using ten digits (*$0, 1, 2, 3, \ldots 8, 9$*) to write the same integers.*

**3.6.4** *This —* ***in equal measure*** *— also applies to writing* ***non-integers****.*

## 4. Comparison of two infinite sets

**4.1.1** There is a certain infinite set of irrational (positive) numbers, for example, such as

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \ldots$$

**4.1.2** Let us write down in the form of rows the non-integer part of each infinite irrational number indicated above; in doing so, we limit ourselves to 20 digits. Let us construct Table 4 (see Appendix).

**4.2.1** Let us examine an infinite linear sequence of digits in the record of one of the numbers in Table 4, for example, the non-integer part of the number $\sqrt{10}$

$$0.1622776601\ldots$$

**4.2.2 Let us correspond this infinite set (infinite linear sequence) of digits with the infinite set of all natural numbers:**

| | | | |
|---|---|---|---|
| 1st digit of the non-integer part of the number $\sqrt{10}$ (digit 1), | 2nd digit (6), | 3rd digit (2), | 4th digit (2), ... |
| **number 1**, | **number 2**, | **number 3**, | **number 4**, ... |

**4.2.3** Let us write down the natural numbers sequentially:

$1, 2, 3, 4, 5, 6, \ldots$ — these are *cardinal* numbers;

1st, 2nd, 3rd, 4th, 5th, 6th, ... — these are *ordinal* numbers ("enumerative").

**4.2.4** According to Cantor, the infinite set of all natural numbers has cardinality $\aleph_0$ (**$\aleph_0$ is the cardinal transfinite number** corresponding to this set).

**4.2.5** Let us present what was stated above in §4.2.2 in another way:

| | | | |
|---|---|---|---|
| 1st digit, | 2nd digit, | 3rd digit, | 4th digit, ... |
| **number 1**, | **number 2**, | **number 3**, | **number 4**, ... |

**4.2.6 We state:**

- **in both rows, the same infinite set of all natural numbers is manifested**;
- *in the first row, natural numbers are manifested as* ***ordinals*** *(digits are counted), and in the second row, these numbers are* ***cardinals****.*

**4.2.7** It follows that **the row** considered in §4.2.1, §4.2.2 **is a countable infinite set (infinite linear sequence) of digits, which *has cardinality* $\aleph_0$** *(this applies* ***to all rows*** *of Table 4).*

# 5. Conclusion<br>The exponent $\aleph_0$ and the formulas of the continuum

**5.1.1** Let us present formulas reflecting the two number systems considered in Sections 2-4.

$$\mathbf{2^{\aleph_0}} \text{ and } \mathbf{10^{\aleph_0}}$$

**5.1.2** As follows from Section 4 (§4.2.3–§4.2.7), **the transfinite cardinal number** — ***the exponent* $\aleph_0$** — *expresses the* ***maximum*** *number of digits in recording a real number, i.e., indicates the cardinality of the corresponding infinite set (infinite linear sequence) of digits in recording a number.*

**5.1.3 This — *in equal measure* — applies to the number of digits in recording a number in the binary system (with zero and one) and in the decimal system (with digits $0, 1, 2, 3, \ldots, 7, 8, 9$). *The linear sequence of digits*** *in recording a real number*, **if it is infinite, *then this sequence (infinite set) of digits* — in both systems — *has the same cardinality; this cardinality is* $\aleph_0$.**

**5.1.4** Let us present 20 rows (20 numbers) **from Table 4**, but only non-integer parts of the numbers.

0.41421356237309504880...
0.73205080756887729352...
0.23606797749978969640...
0.44948974278317809819...
0.64575131106459059050...
0.82842712474619009760...
.............................
0.79583152331271954159...
0.89897948556635619639...
.............................

**Each row above is an infinite linear sequence of digits in the decimal system, having cardinality $\aleph_0$.**

**5.2.1** Let us present two more formulas

$$2^{\aleph_0} = \bar{\mathbf{c}}; \qquad 10^{\aleph_0} = \bar{\mathbf{c}}$$

(the symbol $\bar{\mathbf{c}}$ denotes the continuum of (positive) real numbers).

**5.2.2 The first formula (*currently accepted*) is incorrect:** *the binary number system does not describe and cannot describe the continuum of real numbers.*

**5.2.3 Comparing the two formulas given in §5.2.1, we conclude:**

> *since the exponent in both formulas is the same, the continuum of (positive) real numbers can only be displayed by the second formula (see below, §5.2.4).*

**5.2.4** What is presented in §5.1.1–§5.2.3 means that

> ***the formula*** $\mathbf{10^{\aleph_0}}$ *describes continua of real numbers in the intervals from zero to one, from 1 to 2, from 2 to 3, ...*

**5.3.1** Let us use the same 20 rows from Table 4, but write them **in another way:** ***in the form of an integer part and a non-integer part of the number*** (*these changes in the representation of numbers are made conditionally and only conditionally*)

$$\begin{array}{l} \mathbf{4}, 1421356237309504880\ldots \\ \mathbf{73}, 205080756887729352\ldots \\ \mathbf{236}, 06797749978969640\ldots \\ \mathbf{4494}, 8974278317809819\ldots \\ \mathbf{64575}, 131106459059050\ldots \\ \mathbf{828427}, 12474619009760\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

**5.3.2** The numbers presented (*written conditionally*) show the following:

> ***the continuum of all (positive) real numbers is expressed by the formula***

$$\boldsymbol{\aleph_0(10^{\aleph_0}) = \hat{c}}$$

(when using the decimal number system to represent real numbers).

## References


[1] Paul J. Cohen. *Set theory and the continum hypothesis*. New York, W.A. Benjamin, 1966, p. 163.

$\sqrt{2} = \mathbf{1.41421356237}\ldots\ldots$

$$4 \xrightarrow[\textbf{first row}]{\textbf{40 digits}} 6$$

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 4 | 2 | 1 | 3 | 5 | 6 | 2 | 3 | 7 | 3 | 0 | 9 | 5 | 0 | 4 | 8 | 8 | 0 | 1 | 6 | 8 | 8 | 7 | 2 | 4 | 2 | 0 | 9 | 6 | 9 | 8 | 0 | 7 | 8 | 5 | 6 | 9 | 6 |
| 7 | 1 | 8 | 7 | 5 | 3 | 7 | 6 | 9 | 4 | 8 | 0 | 7 | 3 | 1 | 7 | 6 | 6 | 7 | 9 | 7 | 3 | 7 | 9 | 9 | 0 | 7 | 3 | 2 | 4 | 7 | 8 | 4 | 6 | 2 | 1 | 0 | 7 | 0 | 3 |
| 8 | 8 | 5 | 0 | 3 | 8 | 7 | 5 | 3 | 4 | 3 | 2 | 7 | 6 | 4 | 1 | 5 | 7 | 2 | 7 | 3 | 5 | 0 | 1 | 3 | 8 | 4 | 6 | 2 | 3 | 0 | 9 | 1 | 2 | 2 | 9 | 7 | 0 | 2 | 4 |
| 9 | 2 | 4 | 8 | 3 | 6 | 0 | 5 | 5 | 8 | 5 | 0 | 7 | 3 | 7 | 2 | 1 | 2 | 6 | 4 | 4 | 1 | 2 | 1 | 4 | 9 | 7 | 0 | 9 | 9 | 9 | 3 | 5 | 8 | 3 | 1 | 4 | 1 | 3 | 2 |
| 2 | 2 | 6 | 6 | 5 | 9 | 2 | 7 | 5 | 0 | 5 | 5 | 9 | 2 | 7 | 5 | 5 | 7 | 9 | 9 | 9 | 5 | 0 | 5 | 0 | 1 | 1 | 5 | 2 | 7 | 8 | 2 | 0 | 6 | 0 | 5 | 7 | 1 | 4 | 7 |
| 0 | 1 | 0 | 9 | 5 | 5 | 9 | 9 | 7 | 1 | 6 | 0 | 5 | 9 | 7 | 0 | 2 | 7 | 4 | 5 | 3 | 4 | 5 | 9 | 6 | 8 | 6 | 2 | 0 | 1 | 4 | 7 | 2 | 8 | 5 | 1 | 7 | 4 | 1 | 8 |
| 6 | 4 | 0 | 8 | 8 | 9 | 1 | 9 | 8 | 6 | 0 | 9 | 5 | 5 | 2 | 3 | 2 | 9 | 2 | 3 | 0 | 4 | 8 | 4 | 3 | 0 | 8 | 7 | 1 | 4 | 3 | 2 | 1 | 4 | 5 | 0 | 8 | 3 | 9 | 7 |
| 6 | 2 | 6 | 0 | 3 | 6 | 2 | 7 | 9 | 9 | 5 | 2 | 5 | 1 | 4 | 0 | 7 | 9 | 8 | 9 | 6 | 8 | 7 | 2 | 5 | 3 | 3 | 9 | 6 | 5 | 4 | 6 | 3 | 3 | 1 | 8 | 0 | 8 | 8 | 2 |
| 9 | 6 | 4 | 0 | 6 | 2 | 0 | 6 | 1 | 5 | 2 | 5 | 8 | 3 | 5 | 2 | 3 | 9 | 5 | 0 | 5 | 4 | 7 | 4 | 5 | 7 | 5 | 0 | 2 | 8 | 7 | 7 | 5 | 9 | 9 | 6 | 1 | 7 | 2 | 9 |
| 8 | 3 | 5 | 5 | 7 | 5 | 2 | 2 | 0 | 3 | 3 | 7 | 5 | 3 | 1 | 8 | 5 | 7 | 0 | 1 | 1 | 3 | 5 | 4 | 3 | 7 | 4 | 6 | 0 | 3 | 4 | 0 | 8 | 4 | 9 | 8 | 8 | 4 | 7 | 1 |

Table 1

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | .... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 4 | 1 | 4 | 2 | 1 | 3 | 5 | 6 | 2 | 3 | 7 | 3 | 0 | 9 | 5 | 0 | 4 | 8 | 8 | 0 | .... |
| **2** | 1 | 6 | 8 | 8 | 7 | 2 | 4 | 2 | 0 | 9 | 6 | 9 | 8 | 0 | 7 | 8 | 5 | 6 | 9 | 6 | .... |
| **3** | 7 | 1 | 8 | 7 | 5 | 3 | 7 | 6 | 9 | 4 | 8 | 0 | 7 | 3 | 1 | 7 | 6 | 6 | 7 | 9 | .... |
| **4** | 7 | 3 | 7 | 9 | 9 | 0 | 7 | 3 | 2 | 4 | 7 | 8 | 4 | 6 | 2 | 1 | 0 | 7 | 0 | 3 | .... |
| **5** | 8 | 8 | 5 | 0 | 3 | 8 | 7 | 5 | 3 | 4 | 3 | 2 | 7 | 6 | 4 | 1 | 5 | 7 | 2 | 7 | .... |
| **6** | 3 | 5 | 0 | 1 | 3 | 8 | 4 | 6 | 2 | 3 | 0 | 9 | 1 | 2 | 2 | 9 | 7 | 0 | 2 | 4 | .... |
| **7** | 9 | 2 | 4 | 8 | 3 | 6 | 0 | 5 | 5 | 8 | 5 | 0 | 7 | 3 | 7 | 2 | 1 | 2 | 6 | 4 | .... |
| **8** | 4 | 1 | 2 | 1 | 4 | 9 | 7 | 0 | 9 | 9 | 9 | 3 | 5 | 8 | 3 | 1 | 4 | 1 | 3 | 2 | .... |
| **9** | 2 | 2 | 6 | 6 | 5 | 9 | 2 | 7 | 5 | 0 | 5 | 5 | 9 | 2 | 7 | 5 | 5 | 7 | 9 | 9 | .... |
| **10** | 9 | 5 | 0 | 5 | 0 | 1 | 1 | 5 | 2 | 7 | 8 | 2 | 0 | 6 | 0 | 5 | 7 | 1 | 4 | 7 | .... |
| **11** | 0 | 1 | 0 | 9 | 5 | 5 | 9 | 9 | 7 | 1 | 6 | 0 | 5 | 9 | 7 | 0 | 2 | 7 | 4 | 5 | .... |
| **12** | 3 | 4 | 5 | 9 | 6 | 8 | 6 | 2 | 0 | 1 | 4 | 7 | 2 | 8 | 5 | 1 | 7 | 4 | 1 | 8 | .... |
| **13** | 6 | 4 | 0 | 8 | 8 | 9 | 1 | 9 | 8 | 6 | 0 | 9 | 5 | 5 | 2 | 3 | 2 | 9 | 2 | 3 | .... |
| **14** | 0 | 4 | 8 | 4 | 3 | 0 | 8 | 7 | 1 | 4 | 3 | 2 | 1 | 4 | 5 | 0 | 8 | 3 | 9 | 7 | .... |
| **15** | 6 | 2 | 6 | 0 | 3 | 6 | 2 | 7 | 9 | 9 | 5 | 2 | 5 | 1 | 4 | 0 | 7 | 9 | 8 | 9 | .... |
| **16** | 6 | 8 | 7 | 2 | 5 | 3 | 3 | 9 | 6 | 5 | 4 | 6 | 3 | 3 | 1 | 8 | 0 | 8 | 8 | 2 | .... |
| **17** | 9 | 6 | 4 | 0 | 6 | 2 | 0 | 6 | 1 | 5 | 2 | 5 | 8 | 3 | 5 | 2 | 3 | 9 | 5 | 0 | .... |
| **18** | 5 | 4 | 7 | 4 | 5 | 7 | 5 | 0 | 2 | 8 | 7 | 7 | 5 | 9 | 9 | 6 | 1 | 7 | 2 | 9 | .... |
| **19** | 8 | 3 | 5 | 5 | 7 | 5 | 2 | 2 | 0 | 3 | 3 | 7 | 5 | 3 | 1 | 8 | 5 | 7 | 0 | 1 | .... |
| **20** | 1 | 3 | 5 | 4 | 3 | 7 | 4 | 6 | 0 | 3 | 4 | 0 | 8 | 4 | 9 | 8 | 8 | 4 | 7 | 1 | .... |
| | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... | .... |

Table 2

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_{1,1}$ | $a_{1,2}$ | $a_{1,3}$ | $a_{1,4}$ | $a_{1,5}$ | $a_{1,6}$ | $a_{1,7}$ | $a_{1,8}$ | $a_{1,9}$ | $a_{1,10}$ | $\cdots$ |
| $a_{2,1}$ | $a_{2,2}$ | $a_{2,3}$ | $a_{2,4}$ | $a_{2,5}$ | $a_{2,6}$ | $a_{2,7}$ | $a_{2,8}$ | $a_{2,9}$ | $a_{2,10}$ | $\cdots$ |
| $a_{3,1}$ | $a_{3,2}$ | $a_{3,3}$ | $a_{3,4}$ | $a_{3,5}$ | $a_{3,6}$ | $a_{3,7}$ | $a_{3,8}$ | $a_{3,9}$ | $a_{3,10}$ | $\cdots$ |
| $a_{4,1}$ | $a_{4,2}$ | $a_{4,3}$ | $a_{4,4}$ | $a_{4,5}$ | $a_{4,6}$ | $a_{4,7}$ | $a_{4,8}$ | $a_{4,9}$ | $a_{4,10}$ | $\cdots$ |
| $a_{5,1}$ | $a_{5,2}$ | $a_{5,3}$ | $a_{5,4}$ | $a_{5,5}$ | $a_{5,6}$ | $a_{5,7}$ | $a_{5,8}$ | $a_{5,9}$ | $a_{5,10}$ | $\cdots$ |
| $a_{6,1}$ | $a_{6,2}$ | $a_{6,3}$ | $a_{6,4}$ | $a_{6,5}$ | $a_{6,6}$ | $a_{6,7}$ | $a_{6,8}$ | $a_{6,9}$ | $a_{6,10}$ | $\cdots$ |
| $a_{7,1}$ | $a_{7,2}$ | $a_{7,3}$ | $a_{7,4}$ | $a_{7,5}$ | $a_{7,6}$ | $a_{7,7}$ | $a_{7,8}$ | $a_{7,9}$ | $a_{7,10}$ | $\cdots$ |
| $a_{8,1}$ | $a_{8,2}$ | $a_{8,3}$ | $a_{8,4}$ | $a_{8,5}$ | $a_{8,6}$ | $a_{8,7}$ | $a_{8,8}$ | $a_{8,9}$ | $a_{8,10}$ | $\cdots$ |
| $a_{9,1}$ | $a_{9,2}$ | $a_{9,3}$ | $a_{9,4}$ | $a_{9,5}$ | $a_{9,6}$ | $a_{9,7}$ | $a_{9,8}$ | $a_{9,9}$ | $a_{9,10}$ | $\cdots$ |
| $a_{10,1}$ | $a_{10,2}$ | $a_{10,3}$ | $a_{10,4}$ | $a_{10,5}$ | $a_{10,6}$ | $a_{10,7}$ | $a_{10,8}$ | $a_{10,9}$ | $a_{10,10}$ | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

Table 3

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| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | …. | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **4** | 1 | 4 | 2 | 1 | 3 | 5 | 6 | 2 | 3 | 7 | 3 | 0 | 9 | 5 | 0 | 4 | 8 | 8 | 0 | …. | **√2** | = | 1.41421356237309504880... |
| 2 | 7 | **3** | 2 | 0 | 5 | 0 | 8 | 0 | 7 | 5 | 6 | 8 | 8 | 7 | 7 | 2 | 9 | 3 | 5 | 2 | …. | **√3** | = | 1.73205080756887729352... |
| 3 | 2 | 3 | **6** | 0 | 6 | 7 | 9 | 7 | 7 | 4 | 9 | 9 | 7 | 8 | 9 | 6 | 9 | 6 | 4 | 0 | …. | **√5** | = | 2.23606797749978969640... |
| 4 | 4 | 4 | 9 | **4** | 8 | 9 | 7 | 4 | 2 | 7 | 8 | 3 | 1 | 7 | 8 | 0 | 9 | 8 | 1 | 9 | …. | **√6** | = | 2.44948974278317809819... |
| 5 | 6 | 4 | 5 | 7 | **5** | 1 | 3 | 1 | 1 | 0 | 6 | 4 | 5 | 9 | 0 | 5 | 9 | 0 | 5 | 0 | …. | **√7** | = | 2.64575131106459059050... |
| 6 | 8 | 2 | 8 | 4 | 2 | **7** | 1 | 2 | 4 | 7 | 4 | 6 | 1 | 9 | 0 | 0 | 9 | 7 | 6 | 0 | …. | **√8** | = | 2.82842712474619009760... |
| 7 | 1 | 6 | 2 | 2 | 7 | 7 | **6** | 6 | 0 | 1 | 6 | 8 | 3 | 7 | 9 | 3 | 3 | 1 | 9 | 9 | …. | **√10** | = | 3.16227766016837933199... |
| 8 | 3 | 1 | 6 | 6 | 2 | 4 | 7 | **9** | 0 | 3 | 5 | 5 | 3 | 9 | 9 | 8 | 4 | 9 | 1 | 1 | …. | **√11** | = | 3.31662479035539984911... |
| 9 | 4 | 6 | 4 | 1 | 0 | 1 | 6 | 1 | **5** | 1 | 3 | 7 | 7 | 5 | 4 | 5 | 8 | 7 | 0 | 5 | …. | **√12** | = | 3.46410161513775458705... |
| 10 | 6 | 0 | 5 | 5 | 5 | 1 | 2 | 7 | 5 | **4** | 6 | 3 | 9 | 8 | 9 | 2 | 9 | 3 | 1 | 1 | …. | **√13** | = | 3.60555127546398929311... |
| 11 | 7 | 4 | 1 | 6 | 5 | 7 | 3 | 8 | 6 | 7 | **7** | 3 | 9 | 4 | 1 | 3 | 8 | 5 | 5 | 8 | …. | **√14** | = | 3.74165738677394138558... |
| 12 | 8 | 7 | 2 | 9 | 8 | 3 | 3 | 4 | 6 | 2 | 0 | **7** | 4 | 1 | 6 | 8 | 8 | 5 | 1 | 7 | …. | **√15** | = | 3.87298334620741688517... |
| 13 | 1 | 2 | 3 | 1 | 0 | 5 | 6 | 2 | 5 | 6 | 1 | 7 | **6** | 6 | 0 | 5 | 4 | 9 | 8 | 2 | …. | **√17** | = | 4.12310562561766054982... |
| 14 | 2 | 4 | 2 | 6 | 4 | 0 | 6 | 8 | 7 | 1 | 1 | 9 | 2 | **8** | 5 | 1 | 4 | 6 | 4 | 0 | …. | **√18** | = | 4.24264068711928514640... |
| 15 | 3 | 5 | 8 | 8 | 9 | 8 | 9 | 4 | 3 | 5 | 4 | 0 | 6 | 7 | **3** | 5 | 5 | 2 | 2 | 3 | …. | **√19** | = | 4.35889894354067355223... |
| 16 | 4 | 7 | 2 | 1 | 3 | 5 | 9 | 5 | 4 | 9 | 9 | 9 | 5 | 7 | 9 | **3** | 9 | 2 | 8 | 1 | …. | **√20** | = | 4.47213595499957939281... |
| 17 | 5 | 8 | 2 | 5 | 7 | 5 | 6 | 9 | 4 | 9 | 5 | 5 | 8 | 4 | 0 | 0 | **0** | 6 | 5 | 8 | …. | **√21** | = | 4.58257569495584000658... |
| 18 | 6 | 9 | 0 | 4 | 1 | 5 | 7 | 5 | 9 | 8 | 2 | 3 | 4 | 2 | 9 | 5 | 5 | **4** | 5 | 6 | …. | **√22** | = | 4.69041575982342955456... |
| 19 | 7 | 9 | 5 | 8 | 3 | 1 | 5 | 2 | 3 | 3 | 1 | 2 | 7 | 1 | 9 | 5 | 4 | 1 | **5** | 9 | …. | **√23** | = | 4.79583152331271954159... |
| 20 | 8 | 9 | 8 | 9 | 7 | 9 | 4 | 8 | 5 | 5 | 6 | 6 | 3 | 5 | 6 | 1 | 9 | 6 | 3 | **9** | …. | **√24** | = | 4.89897948556635619639... |
| | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | …. | | | |

Table 4